Episode 13

Free Vibration of Damped Systems

ENGN0040: Dynamics and Vibrations
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Topics for todays class

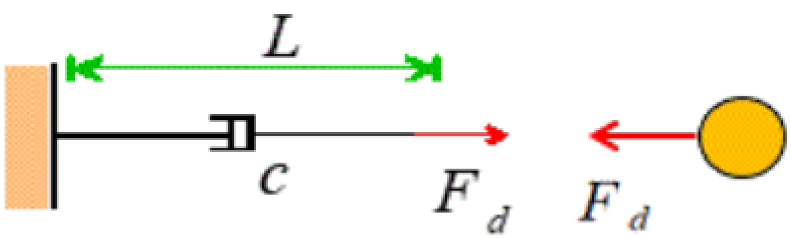
Free Vibration of Damped Systems

1. Modeling energy dissipation – the dashpot
2. Damped harmonic oscillator
3. Terminology
4. Examples
5. Measuring damping experimentally

5·5 Free Vibration of damped systems

Goal : Understand influence of energy loss on free vibrations

5·5·1 Modeling energy loss : The dashpot



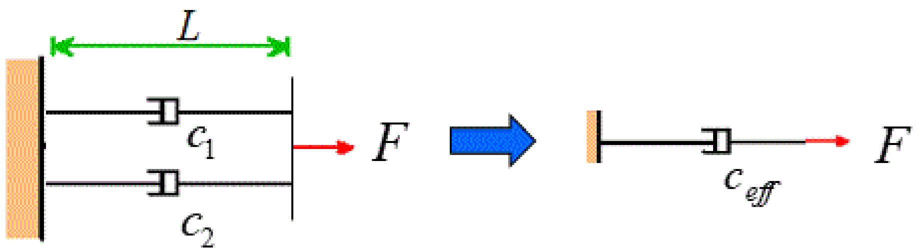
Exerts a force proportional to stretch rate

$$F_d = c \frac{dL}{dt}$$

c : "Dashpot coefficient"
(constant – similar to spring stiffness)

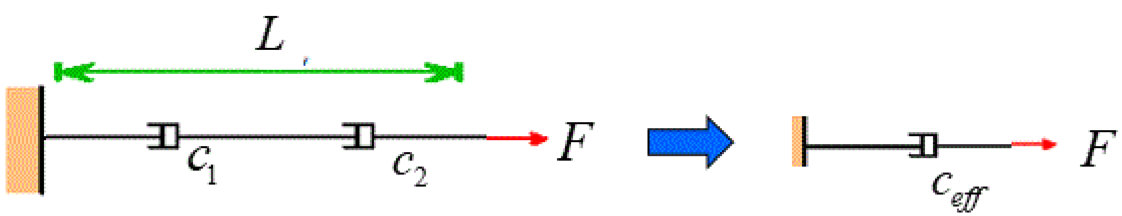
Units   N s/m

Can combine like springs



Parallel

$$c_{eff} = c_1 + c_2$$

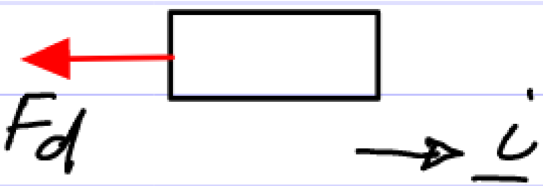


Series

$$\frac{1}{c_{eff}} = \frac{1}{c_1} + \frac{1}{c_2}$$

Power to stretch a dashpot

$$P = \underline{F} \cdot \underline{v} = -c\frac{dL}{dt}\underline{i} \cdot \frac{dL}{dt}\underline{i}$$



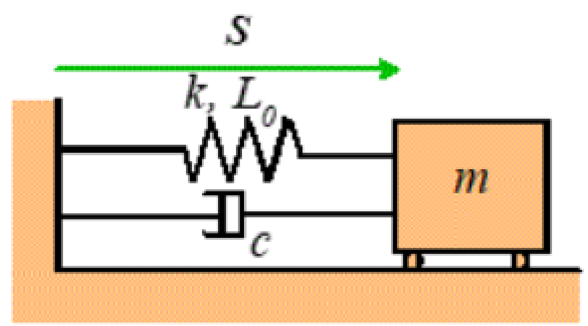
$$P = -c\left(\frac{dL}{dt}\right)^2$$

Always < 0

$\Rightarrow$ always dissipates energy

5.5.2 Free Vibration of a 1-DOF system

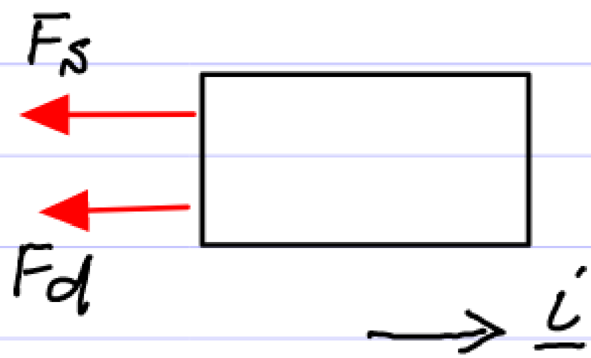
Canonical vibration problem: The spring-mass system is released with speed v_0 from position s_0 at time $t=0$. Find $s(t)$



Approach : (1) EOM

(2) Find solution in tables

EOM ($\underline{F} = m\underline{a}$)



$$-F_s - F_d = m\frac{d^2s}{dt^2}$$

$$F_s = k(s - L_0) \qquad F_d = c\,ds/dt$$

$$-k(s - L_0) - c\,ds/dt = m\,d^2s/dt^2$$

$$\Rightarrow \quad \boxed{\frac{m}{k}\frac{d^2s}{dt^2} + \frac{c}{k}\frac{ds}{dt} + s = L_0}$$

List of standard ODEs for vibration problems

Case I $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$

Case II $\frac{1}{\alpha^2}\frac{d^2x}{dt^2}-x=-C$

Case III $\boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C}$

Case IV $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF(t)$ with $F(t)=F_0\sin\omega t$

Case V $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$ with $y(t)=Y_0\sin\omega t$

Case VI $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$ with $y(t)=Y_0\sin\omega t$

Case VII $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right)$ with $y(t)=Y_0\sin\omega t$

Our eq: $\frac{m}{k}\frac{d^2s}{dt^2}+\frac{c}{k}\frac{ds}{dt}+s=L_0$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C$$

$x \doteq s$

$C \doteq L_0$

$\frac{1}{\omega_n^2} \doteq \frac{m}{k}$

$\frac{2\zeta}{\omega_n} = \frac{c}{k}$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

$$\boxed{\zeta = \frac{c}{2\sqrt{km}} \quad \text{“Damping Coefficient”}}$$

Solution to Case III (From pdf on website)

Solution to $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$ with $x=x_0$ $\frac{dx}{dt}=v_0$ $t=0$

Let $\omega_d=\omega_n\sqrt{\left|1-\varsigma^2\right|}$ (Note absolute value)

Overdamped $\varsigma>1$ $x(t)=C+\exp(-\varsigma\omega_n t)\left\{\frac{v_0+(\varsigma\omega_n+\omega_d)(x_0-C)}{2\omega_d}\exp(\omega_d t)-\frac{v_0+(\varsigma\omega_n-\omega_d)(x_0-C)}{2\omega_d}\exp(-\omega_d t)\right\}$

Critically Damped $\varsigma=1$ $x(t)=C+\left\{(x_0-C)+\left[v_0+\omega_n(x_0-C)\right]t\right\}\exp(-\omega_n t)$

Underdamped $\varsigma<1$ $x(t)=C+\exp(-\varsigma\omega_n t)\left\{(x_0-C)\cos\omega_d t+\frac{v_0+\varsigma\omega_n(x_0-C)}{\omega_d}\sin\omega_d t\right\}$

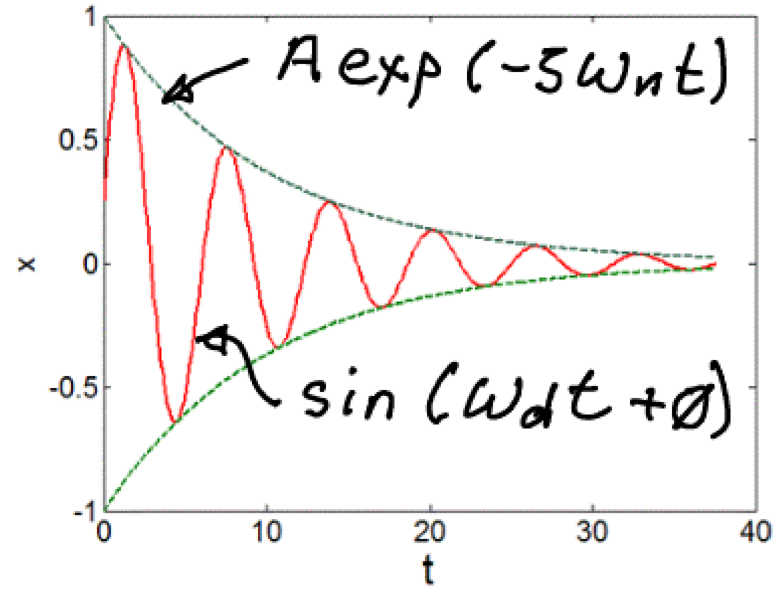
Recall $\omega_n=\sqrt{\frac{k}{m}}$

$\zeta=\frac{c}{2\sqrt{km}}$

Underdamped Solution $(\zeta < 1)$

$$x(t) = A \exp(-\zeta \omega_n t) \sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$



Notes:

(1) Exponentially decaying vibrations

(2) Frequency ω_d (slightly smaller than ω_n)

(3) Decays as $\exp(-\zeta \omega_n t)$

Vibrations decay faster as $\zeta \to 1$

$\boxed{\text{Fastest decay for } \zeta = 1 \text{ - Critical Damping}}$

Overdamped Solution $\zeta > 1$

$$x(t) = \exp(-\zeta\omega_n t)\left\{A\exp(\omega_d t) + B\exp(-\omega_d t)\right\}$$

$$\omega_d = \omega_n\sqrt{\zeta^2 - 1}$$

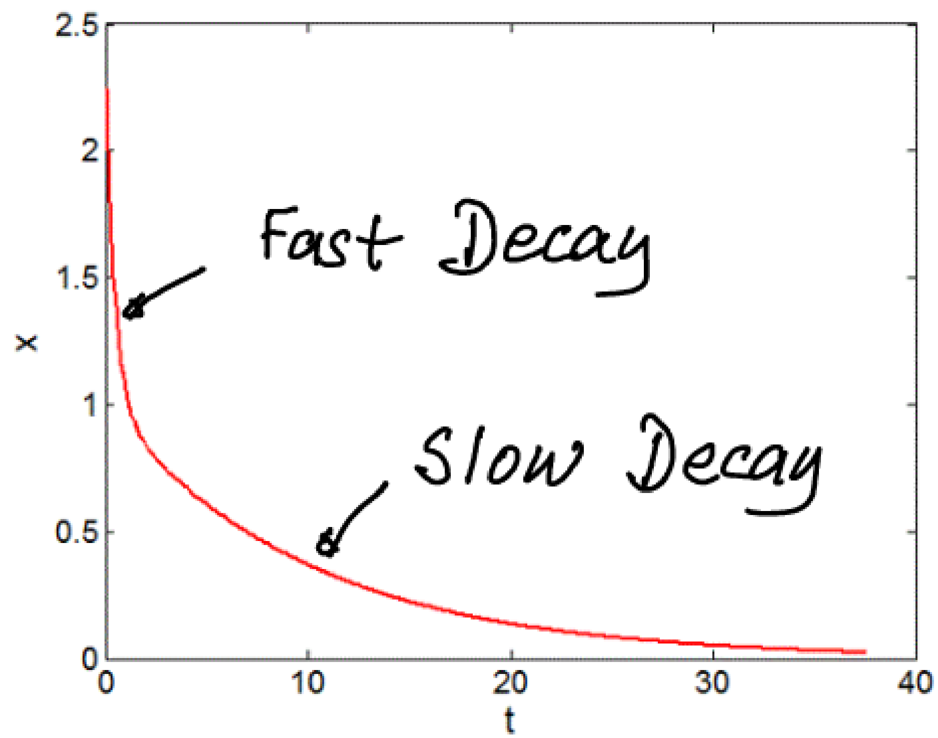
$$\Rightarrow x(t) = \underbrace{A\exp\left\{-\omega_n t\left(\zeta - \sqrt{\zeta^2-1}\right)\right\}}_{\text{Decays slowly}} + \underbrace{B\exp\left\{-\omega_n t\left(\zeta + \sqrt{\zeta^2-1}\right)\right\}}_{\text{Decays quickly}}$$

Notes:

(1) No vibrations

(2) Slowest term decays faster as $\zeta \to 1$
slower as $\zeta \to \infty$

$$\boxed{\text{Fastest decay for } \zeta = 1}$$



Engineering Implications

(1) To stop vibrations $\zeta \geq 1 \Rightarrow c \geq 2\sqrt{km}$

(2) For fastest return to equilibrium

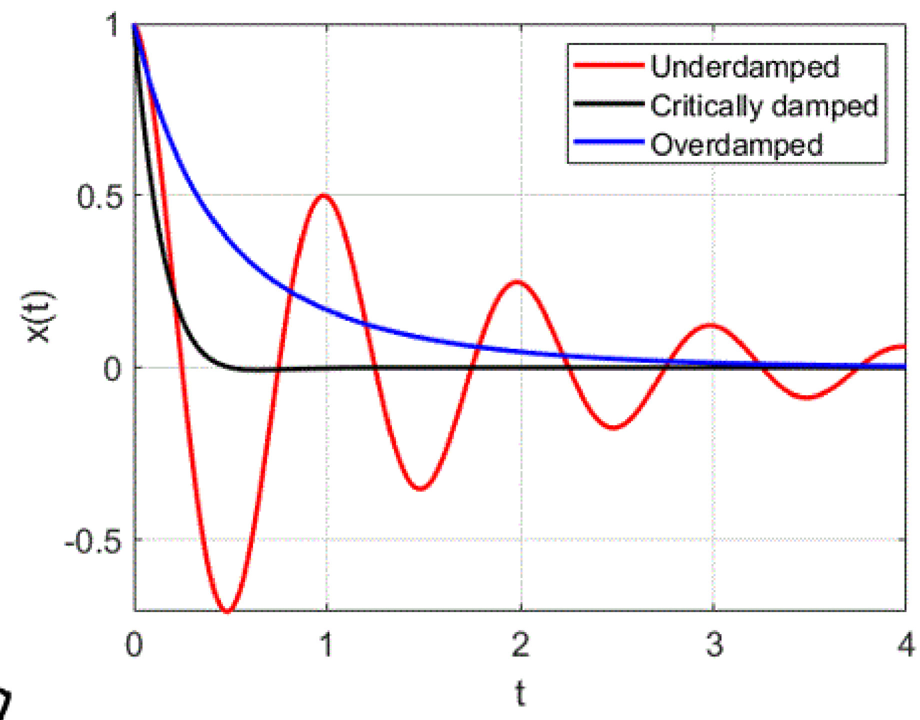
(a) Choose critical damping $\zeta = 1$

Then $x \approx A \exp(-\omega_n t)$

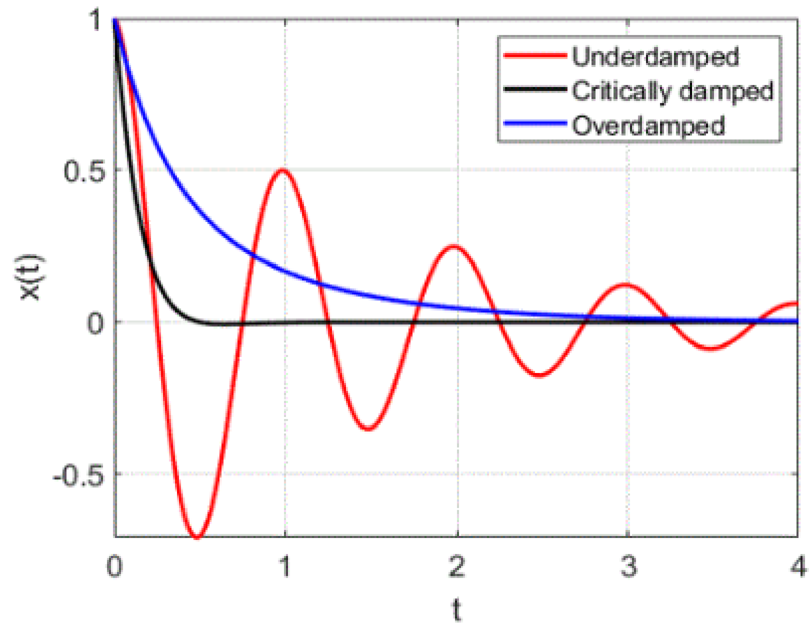
Hence maximize ω_n

$\Rightarrow$ maximize $\omega_n = \sqrt{\frac{k}{m}}$

Choose $c = 2\sqrt{km}$



Underdamped System



Amplitude decays as $\exp(-\zeta \omega_n t)$

$\Rightarrow$ increase $\zeta \omega_n = c/(2m)$

$\Rightarrow$ increase c or reduce m

Overdamped System

x decays as $\exp\{-\omega_n(\zeta - \sqrt{\zeta^2 - 1})t\}$

$\approx \exp\{-\omega_n t/(2\zeta)\}$ for $\zeta \gg 1$

$\Rightarrow$ increase $\omega_n/(2\zeta) = k/(2c)$

$\Rightarrow$ increase k or reduce c

Solving the case III vibration equation

Background: Complex Variables

Define $i=\sqrt{-1}$ General complex number $z=a+ib$

Complex conjugate $\bar{z}=a-ib$ $\Rightarrow$ $a=(z+\bar{z})/2 \quad b=-i(z-\bar{z})/2$

Euler's formula $e^{i\theta}=\cos\theta+i\sin\theta$

Polar / rectangular conversion

$$a+ib=\rho e^{i\theta} \quad \rho=\sqrt{a^2+b^2} \quad \theta=\tan^{-1}(b/a)$$

$$a=\rho\cos\theta \quad b=\rho\sin\theta$$

Trig functions

$$\cos\theta=\left(e^{i\theta}+e^{-i\theta}\right)/2$$

$$\sin\theta=-i\left(e^{i\theta}-e^{-i\theta}\right)/2$$

Calculus $\dfrac{de^{i\omega t}}{dt}=i\omega e^{i\omega t}$ $\dfrac{d^2e^{i\omega t}}{dt^2}=-\omega^2 e^{i\omega t}$

Solving the case III vibration equation

Solve: $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C \qquad x=x_0 \qquad \frac{dx}{dt}=v_0 \qquad t=0$

Guess $x=Ae^{\lambda t}+C \longrightarrow \left(\frac{\lambda^2}{\omega_n^2}+\frac{2\zeta\lambda}{\omega_n}+1\right)Ae^{\lambda t}=0$ (characteristic equation)

Roots $\lambda=-\zeta\omega_n \mp \omega_n\sqrt{\zeta^2-1}$

$\zeta>1$ (Overdamped) $\Rightarrow$ two real roots $\qquad \lambda=-\zeta\omega_n \pm \omega_d$

$\zeta=1$ (Critically damped) $\Rightarrow$ one real root $\qquad \lambda=-\omega_n$

$\zeta<1$ (Underdamped) $\Rightarrow$ two complex roots $\qquad \lambda=-\zeta\omega_n \pm i\omega_d$

$\omega_d=\sqrt{|\zeta^2-1|}$

Note absolute value

General Solution:

$$\zeta>1 \Rightarrow x=C+A_1e^{(-\zeta\omega_n+\omega_d)t}+A_2e^{(-\zeta\omega_n-\omega_d)t}$$

$$\zeta=1 \Rightarrow x=C+A_1e^{-\zeta\omega_n t}+A_2te^{-\zeta\omega_n t}$$

$$\zeta<1 \Rightarrow x=C+A_1e^{(-\zeta\omega_n+i\omega_d)t}+A_2e^{(-\zeta\omega_n-i\omega_d)t}$$

A_1, A_2 Determined by initial conditions

Solving the case III vibration equation

Overdamped Solution: $\zeta > 1 \Rightarrow x = C + A_1 e^{(-\zeta\omega_n + \omega_d)t} + A_2 e^{(-\zeta\omega_n - \omega_d)t}$

Initial Conditions: $x(0) = C + A_1 + A_2 = x_0$

$$\left.\frac{dx}{dt}\right|_{t=0} = A_1(\omega_d - \zeta\omega_n) - A_1(\omega_d + \zeta\omega_n) = v_0$$

$$\Rightarrow A_1 = \frac{v_0 + (\zeta\omega_n + \omega_d)(x_0 - C)}{2\omega_d} \quad A_2 = -\frac{v_0 + (\zeta\omega_n - \omega_d)(x_0 - C)}{2\omega_d}$$

$$x(t) = C + \exp(-\zeta\omega_n t)\left\{\frac{v_0 + (\zeta\omega_n + \omega_d)(x_0 - C)}{2\omega_d}\exp(\omega_d t) - \frac{v_0 + (\zeta\omega_n - \omega_d)(x_0 - C)}{2\omega_d}\exp(-\omega_d t)\right\}$$

Solving the case III vibration equation

Critically Damped Solution: $\zeta = 1 \Rightarrow x = C + A_1 e^{-\zeta\omega_n t} + A_2 t e^{-\zeta\omega_n t}$

Initial Conditions:

$$x(0) = C + A_1 = x_0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = -\omega_n A_1 + A_2 = v_0$$

$$A_1 = x_0 - C \quad A_2 = v_0 + \omega_n (x_0 - C)$$

$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n (x_0 - C)\right]t\right\}\exp(-\omega_n t)$$

Solving the case III vibration equation

Underdamped Solution: $\zeta < 1 \Rightarrow x = C + A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_2 e^{(-\zeta\omega_n - i\omega_d)t}$

Initial Conditions: $x(0) = C + A_1 + A_2 = x_0$

$$\left.\frac{dx}{dt}\right|_{t=0} = A_1(i\omega_d - \zeta\omega_n) - A_1(i\omega_d + \zeta\omega_n) = v_0$$

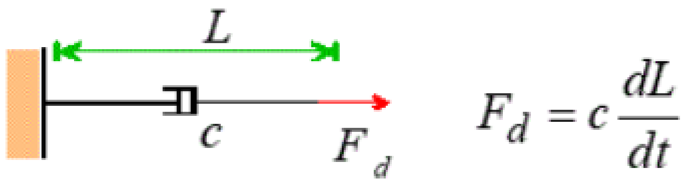
$$\Rightarrow A_1 = -i\frac{v_0 + (\zeta\omega_n + i\omega_d)(x_0 - C)}{2\omega_d} \quad A_2 = i\frac{v_0 + (\zeta\omega_n - i\omega_d)(x_0 - C)}{2\omega_d}$$

$$x(t) = C + \exp(-\zeta\omega_n t)\left\{(x_0 - C)\frac{1}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right) - \frac{v_0 + \zeta\omega_n(x_0 - C)}{\omega_d}\frac{i}{2}\left(e^{i\omega_d t} - e^{-i\omega_d t}\right)\right\}$$

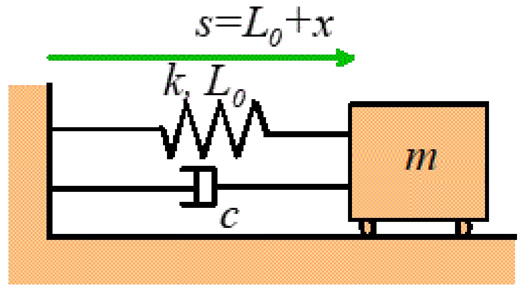
Euler's formula:

$$x(t) = C + \exp(-\zeta\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \zeta\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$

5.5.3 Summary of constants used in damped vibration formulas



(1) The constant c (called "dashpot coefficient" or "damping coefficient"



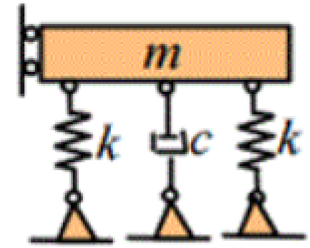
(2) ω_n : "Un-damped natural frequency" $\omega_n = \sqrt{k/m}$

(3) ζ : called "damping coefficient" or "damping ratio" $\zeta = c/(2\sqrt{km})$
Some texts call the product $\zeta\omega_n$ the "damping ratio"

(4) ω_d "Damped natural frequency"
$\omega_d = \omega_n \sqrt{|\zeta^2 - 1|}$

Teminology can be confusing !

5.5.4: Example: The spring-mass system shown in the figure is critically damped.



Find the new damping factor if one spring is removed.

Formula $\zeta = c/(2\sqrt{km})$

"Critical damping" $\Rightarrow$ $\zeta = 1$

For original system $\zeta_1 = \dfrac{c}{2\sqrt{2km}} = 1$

For system with 1 spring removed

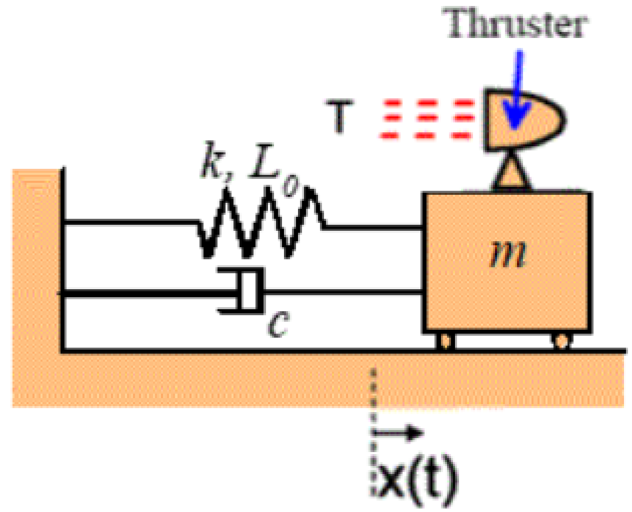
$$\boxed{\zeta_2 = \frac{c}{2\sqrt{km}} = \sqrt{2}\,\zeta_1 = \sqrt{2}}$$

5.5.5 Example: Design a thrust-stand

Constraints:

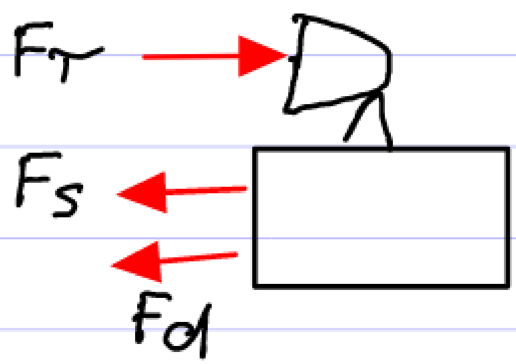
1. Mass of stand+engine 1000kg
2. Max expected thrust: 100 kN
3. Max deflection 1cm
4. Must reach equilibrium deflection as quickly as possible

Find values for *k* and *c*.



Approach: (1) Statics to find k
(2) Critical damping to find c

(1) Statics



$$\Sigma F = 0 \quad \Rightarrow \quad F_T - F_s - F_d = 0$$

$$F_s = kx \qquad F_d = c\,\cancel{\frac{dx}{dt}} = 0$$ because static

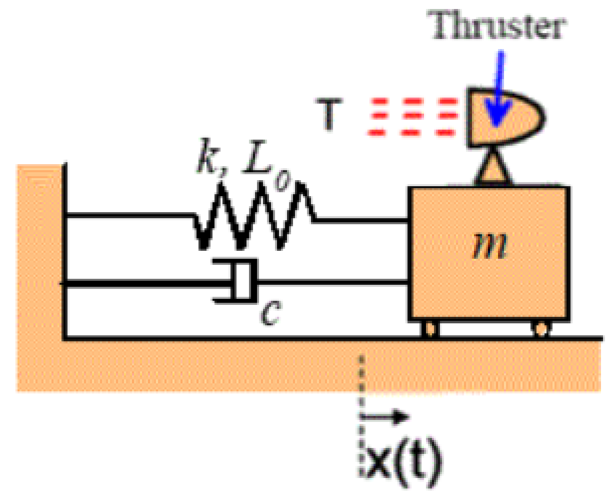
Hence $\boxed{F_T = kx \quad \Rightarrow \quad k = F_T/x = 10^7 \ \mathrm{N/m}}$

5.5.5 **Example**: Design a thrust-stand

Constraints:

1. Mass of stand+engine 1000kg
2. Max expected thrust: 100 kN
3. Max deflection 1cm
4. Must reach equilibrium deflection as quickly as possible

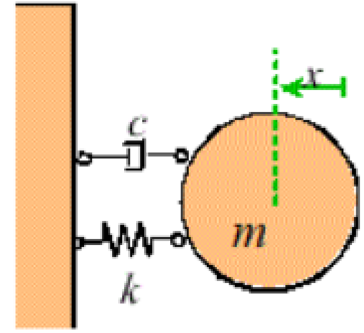
Find values for *k* and *c*.



Critical damping $\Rightarrow \quad \zeta = \dfrac{c}{2\sqrt{km}} = 1$

$$\Rightarrow \quad \boxed{c = 2\sqrt{km} = 200 \text{ kNs/m}}$$

5.5.6: Example: Impact of a baseball is idealized as a damped spring-mass system.



At time *t=0* $x=0,\quad \frac{dx}{dt}=v_0$

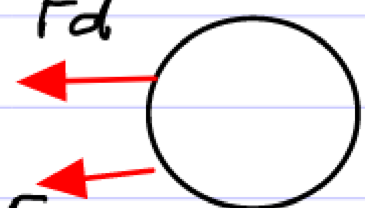
Find a formula for $x(t)$ in terms of ζ, ω_n
If ζ is small, rebound occurs at approximately $x=0$
Show that the restitution coefficient is $e \approx \exp(-\pi\zeta)$

The impact movie shows contact last for 4 millisec and *e=0.6*. The baseball has mass 0.145kg. Find *k,c* for the baseball

Approach (1) Derive EOM ($\underline{F}=m\underline{a}$)
(2) Solve
(3) Use solution to predict rebound velocity $\Rightarrow$ find e

$\underline{F}=m\underline{a}$

$$m\frac{d^2x}{dt^2} = F_s + F_d \qquad F_s = -kx \qquad F_d = -c\frac{dx}{dt}$$



$$\Rightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Hence $\underbrace{\frac{m}{k}}_{1/\omega_n^2} \frac{d^2x}{dt^2} + \underbrace{\frac{c}{k}}_{2\zeta/\omega_n} \frac{dx}{dt} + x = 0$

Case III EOM with $\omega_n = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$

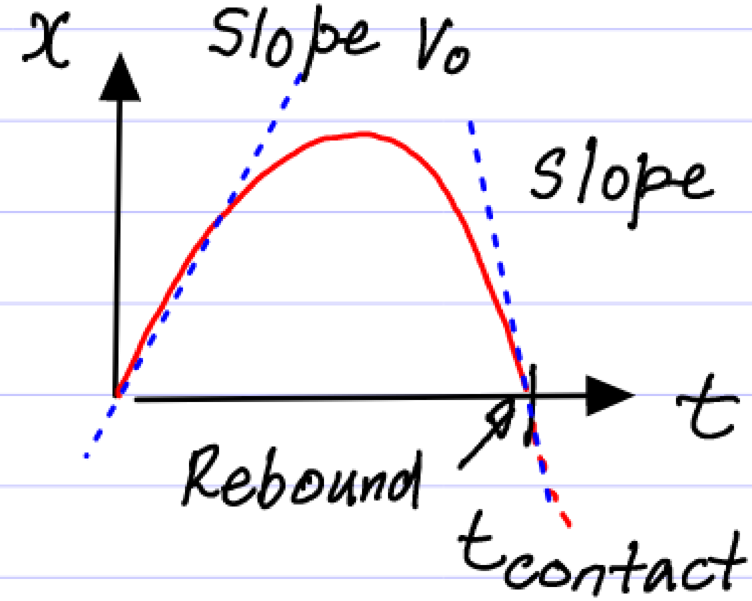
We know ball rebounds $\Rightarrow$ underdamped
$\Rightarrow \zeta < 1$

$$x(t) = C + \exp(-\zeta\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \zeta\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$

$C = 0$ $\quad$ Given $x = 0$ @ $t = 0 \Rightarrow x_0 = 0$
$dx/dt = V_0$ @ $t = 0$

$$\Rightarrow \boxed{x(t) = \frac{V_0}{\omega_d}\exp(-\zeta\omega_n t)\sin\omega_d t \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}}$$

Understanding Solution



Restitution coefficient

$$e = \frac{-V_1}{V_0}$$

At rebound $x \sim 0$

$$\Rightarrow \quad \sin(\omega_d \, t_{contact}) = 0$$

$$\Rightarrow \quad \omega_d \, t_{contact} = \pi$$

$$\Rightarrow \quad t_{contact} = \pi / \omega_d$$

$$V = \frac{dx}{dt} = \frac{d}{dt}\left\{ \frac{V_0}{\omega_d} \exp(-\zeta \omega_n t) \sin \omega_d t \right\}$$

$$= \frac{V_0}{\omega_d} \exp(-\zeta \omega_n t) \left\{ -\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t \right\}$$

Rebound Velocity : substitute $t = \pi/\omega_d$

$$\Rightarrow \quad V_1 = -\frac{V_0}{\cancel{\omega_d}} \exp\left(\frac{-\zeta\omega_n\pi}{\omega_d}\right) \cancel{\omega_d}$$

$$\text{Hence} \quad e = \frac{-V_1}{V_0} = \exp\left(\frac{-\zeta\omega_n\pi}{\omega_d}\right)$$

$\omega_n\sqrt{1-\zeta^2}$

Note for $\zeta \ll 1$ $\quad \sqrt{1-\zeta^2} \approx 1$

$$\text{Hence} \quad \boxed{e = \exp(-\pi\zeta)}$$

Finally find k, c from given data

(1) $e = 0.6 \Rightarrow \exp(-\pi\zeta) = 0.6 \Rightarrow \zeta = \frac{-1}{\pi}\log(0.6)$

$$\Rightarrow \zeta = 0.16$$

(2) $t_{contact} = 4 \times 10^{-3}\,s \qquad t_{contact} = \frac{\pi}{\omega_d}$

$$\Rightarrow \omega_n = \frac{\pi}{t_{contact}\sqrt{1-\zeta^2}} = 7900 \text{ rad/s}$$

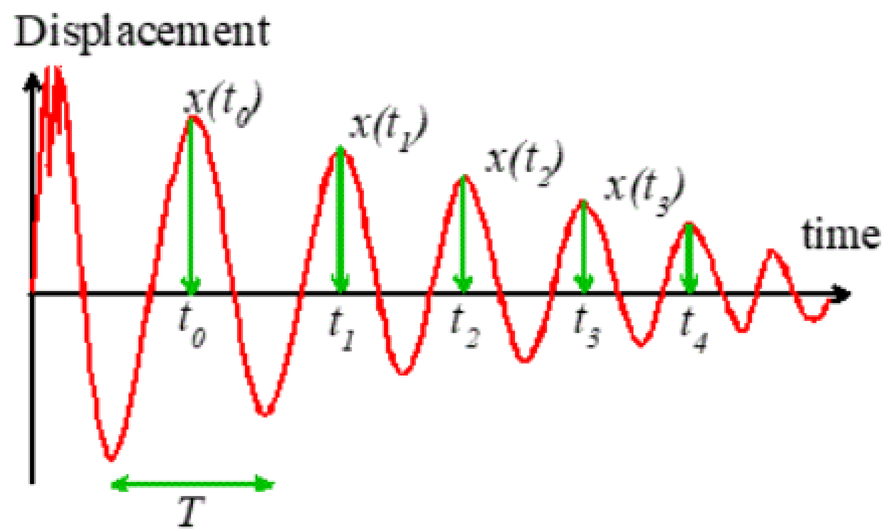
$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 \qquad m = 0.145\,kg$$

$$\Rightarrow \boxed{k = 1.14\,kN/m}$$

$$\zeta = \frac{c}{2\sqrt{km}} \Rightarrow \boxed{c = 2\sqrt{km}\,\zeta = 4.2\ Ns/m}$$

5.5.7 Measuring ω_n and ζ from an impulse test

Impulse test produces a damped vibration response



Procedure

(1) Find period T

(2) Find "log decrement"

$$\delta = \frac{1}{n} \underbrace{\log}_{\text{natural log}} \left(\frac{x(t_0)}{x(t_n)} \right)$$

Then

$$\boxed{\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T}} \qquad \boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}}$$

Proof

Damped vibration sol $x(t) = A\exp(-\zeta\omega_n t)\sin(\omega_d t + \phi)$

Hence $$\frac{x(t_0)}{x(t_n)} = \frac{\cancel{A}\,\cancel{\exp(-\zeta\omega_n t_0)}\ \cancel{\sin(\omega_d t_0 + \phi)}}{\cancel{A}\exp\{-\zeta\omega_n(\cancel{t_0} + nT)\}\ \cancel{\sin\{\omega_d(t_0 + nT) + \phi\}}}$$

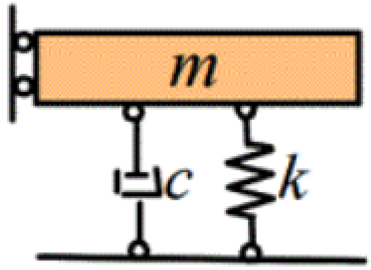
$$\Rightarrow \frac{x(t_0)}{x(t_n)} = \exp(\zeta\omega_n nT)$$

$$\Rightarrow \delta = \frac{1}{n}\log\left(\frac{x(t_0)}{x(t_n)}\right) = \zeta\omega_n T$$

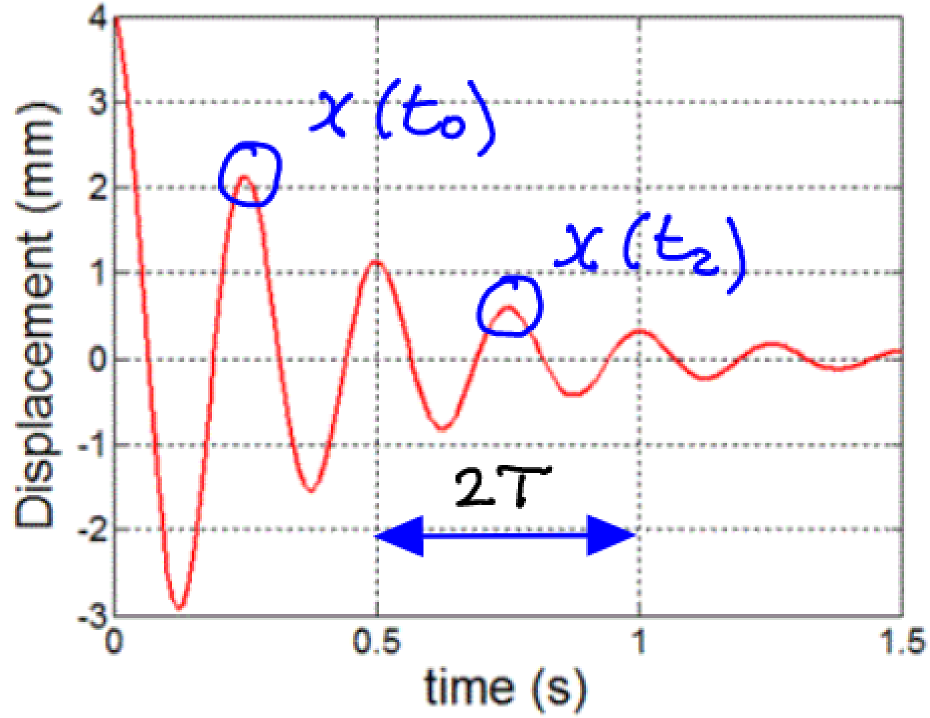
Recall $T = 2\pi/\omega_d$ $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\Rightarrow \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \qquad \omega_n = \frac{\delta}{\zeta T} = \frac{\sqrt{4\pi^2+\delta^2}}{T}$$

Example: Fig shows vibration of a vibration isolation table.



Find ω_n, ζ



Approach: Use formulas

From graph:

(1) $T = 0.25\,s$

(2) Log decrement $\delta = \frac{1}{2} \log\left(\frac{2.1}{0.6}\right) = 0.63$

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.1}$$

$$\boxed{\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} \approx 25 \text{ rad/s}}$$